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Two Person Fair Division of Indivisible Items by Envy Free Algorithms

Arpita Gupta, Jinkal Javia, Sapan Gandhi

Abstract— The problem of fairly dividing the divisible item like a cake, land or a pie goes back to the dawn of civilization. There has been a plethora of solutions given to divide these divisible items among two people. But every property cannot be achieved with the finite number of cuts. Here we deal with the fairly allocation of indivisible items among two people. In this paper two algorithms are presented for the fair division of indivisible items among two players. In both of them both the players have to rank the items starting with the most preferred one to the least preferred one. The job of the players is done here. The algorithms only need this information and no other detail is needed. Hence this is fairly simple algorithm. The algorithms are called BT and AL algorithms. Both algorithms gives envy free allocation of the items among both persons. Same number of items is allocated to both the persons. And the unallocated items are kept in contested pile (CP). Thus in this paper we describe the algorithms and check different scenarios and find out which one is better, which one gives an optimal solution and which one gives a maximal solution.

Index Terms— AL Algortihm, BT Algorithm, Constested Pile, Envy Free Algorithm, Fair Division, Indivisible Items, Pareto-optimal

1 Introduction

THE fair division of divisible items like cake goes back to the Western Literature where the procedure mentioned was 'I cut, you choose' which resulted into fair allocation of the pieces among two people. But the allocations of indivisible item like the division of the property on the event of a divorce or division among the family heirs does not follow the 'cut and choose' concept. Every division cannot be achieved with finite number of cuts [2]. So, for the fair allocation of the indivisible item among two people, two algorithms are introduced.

In both of the algorithms both the players need to rank all items fro m best to worst. Best for them is what they prefer the most and worst is the least one preferred. The first algorithm asks both the players to give their independent choices and descending to the less preferred ones among the unallocated items left. While in the second algorithm requires the complete list should be given to the referee beforehand (or a computer).

The first algorithm was proposed by Brams [3] and Taylor and so it is called BT. It is a 'query step' algorithm for allocating indivisible items fairly among two players. It goes like this: if at any point of time, players A and B name different item the algorithm allocates the items as their preference, otherwise (i.e. if they name same item at same preference) the item goes into the contested pile (CP).

The second algorithm is called AL. Un like BT, it does not allocate all the items to the players but some may go into the contested pile. But in the AL algorithm the number of items in the contested pile may be equal or smaller but never more than that for BT for the same instance.

The similarity between both the algorithms is that both BT and AL when allocates an item to one player, simultaneously allocates another item to another player. And in this way, in both the algorithms, the number of items allocated to both the players remains the same at any time.

The division done by both the algorithms is fair and envy-free (EF): A's items can be matched to the items allocated to B such that A prefers each of its items to each of the items allocated to B and same goes for B i.e . B prefers each of its items over each item allocated to A. Thus it makes the allocation envy-free. It can also be concluded after the following explanation that the allocation done by AL algorithm is maximal i.e. it is not possible to allocate any more items to both the players than already allocated to make the division fair and envy-free. Also, both AL and BT are manipulable: By giving the wrong preferences of their choices the players can end with different items. But in real life situations it is hardly feasible to know the other player's preference list.

2 Envy-Free Allocations

Consider the task of dividing indivisible items among two players, say, A and B. Let the items be { a, b, c, d, e, f }. The task is to divide these items equally and fairly between the players. The definition of envy-freeness needs only the players' preference to know whether the player prefers its own allocated items over the items in the opponent player's set. Let the set of items received by A and B be denoted by S_A and S_B respectively. The allocations are envy free if and only if $|S_A| = |S_B|$. Hence there exists an injection f: $S_A \rightarrow S_{\mathbf{B}}$ and another injection g: $S_B \rightarrow S_{\mathbf{A}}$ such that for each item x received by A, A prefers x to f(x), and for each item y received by B, B prefers y to g(y).

Suppose the players' preferences for the items are as follows:

A: <u>a</u> b <u>c</u> d <u>e</u> f

B: <u>b</u> <u>d</u> <u>f</u> a c e

The underlined are the items allocated to A and B. So, S_A = { a, c, f } and $S_{\mathbf{B}}$ = { b, d, f }. These allocations are envy free. This is because, for A, f (a) = b, f (c) = d, f (e) = f; and for B, g (b) = a, g

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(d) = c, g (f) = e. In short, f (a, c, e) = (b, d, f) and g (b, d, f) = (a, c, e). Thus this allocation is EF. By comparison it can be concluded that the allocation { a, b, c } to A and { d, e , f } to B is not EF.

Thus, an allocation is EF if and only if, for each item x received by A, the number of items received by B that A prefers to x is not greater than the number of items received by A that A prefers to x. The set consisting of A's k most preferred items is equal to the set of B's k most preferred items.

3 BT ALGORITHM

The rules for BT algorithm are:

1. Players A and B name their most preferred item from the unallocated items.
2. If these items are different, items get allocated to the respective players. If they are different they go into the contested pile (CP).
3. If all the items have been allocated to the players or have been gone in the contested pile, then stop. Otherwise, go to step 1.

4 AL ALGORITHM

The rules for AL algorithm are as follows:

In AL algorithm, the whole precedence list is already present. If the same item from beginning is at the same preference then that item is kept in the contested pile. This process is repeated until different items are found at some level of preference. Then those items are allocated. After the first assignment of the items to the players is made, new assignments are made: when the players prefer different items or the players prefer the same item (provided initial assignments are made), the assignment of more preferred item to one player and a less preferred item to the other player does not envy the player, so it can be feasible. Thus, when there is a common preferred item, the feasibility of assigning it to the either player is assessed, one player at a time. If it is not found feasible, the item is placed in the CP.

AL is started with each stage t (t = 0, 1, 2..), exactly t items have been allocated to each player and it goes until there are no unallocated items left. The rules for AL will be clearer with the different illustrations and observations.

5 OBSERVATIONS

<u>Illustration 1:</u>

Let the set of preferences for players A and B be as follows:
A: a b c d
B: b c d a

1. Applying BT

Initially, both the players' first choice is different, so a is allocated to A and b is allocated to B. Next, among unallocated items both A and B's next preference is c, so c goes in CP. Similarly that happens for d. Finally, A receives a, B receives b and CP = { c, d }.

2. Applying AL

Stage 0 allocates a to A and b to B. Proceeding to stage 1, both players prefer c. Consider allocating c to B. Less preferred item for A than c is d. So let's allocate c to B and d to A. But this allocation is not EF because B contains items b and c that are more preferred by A then d. So c cannot be allocated to B. Consider allocating c to A. It means d will be allocated to B. This allocation is feasible because B only prefers one item more than d that is allocated to A. Hence the final allocation is { a, c } to A and { b, d } to B. Here CP is empty. This allocation is EF.

Here using AL, complete allocation is possible. In next example, it can be shown that both BT and AL can produce partial allocations and they may differ.

<u>Illustration 2:</u>

Consider the preferences of players as follows:
A: a b c d e f
B: b c e d a f

1. Applying BT

Initially, a is allocated to A and b is allocated to B. There is a tie for item c, so it goes in CP. Next unallocated preferred items are d and e. So, d is allocated to A and e is allocated to B. Lastly, f goes in CP. So, S_A = { a, d }, S_B = { b, e } and CP = { c, f }.

2. Applying AL

The top ranked items of both the players are different. So, a is allocated to A and b to B. This was stage 0. In stage 1, both players' most preferred item is c. For A, item less preferred than c is d. But c cannot be assigned to B, because then B will be allocated more than one item that A prefers over d that has been allocated to it. So c is assigned to A and e is assigned to B. In stage 2, no further allocation is possible because both d and f are at same preference for both the players. So they go into the CP. Thus, applying AL, S_A = { a, c }, S_B = { b, e } and CP = { d, f }.

By these two observations it can be concluded that the number of items allocated to a player by AL is never less than the number of items allocated by BT. It can be more or equal. If the number of items are same, but the items are different, then the AL allocation Pareto-dominates the BT allocation.

<u>Illustration 3:</u>

Suppose the preferences' sets are as follows:

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A: 1 2 3 4 5 6
B: 2 3 1 5 6 4

1. Applying BT

As the first preference is different for both the players, 1 is allocated to A and 2 is allocated to B. 3 goes in the CP. Next, 4 is allocated to A and 5 is allocated to B. 6 goes in the CP. So, SA = { 1, 4 }, SB = { 2, 5 } and CP = { 3, 6 }.

2. Applying AL

At stage 0, 1 is assigned to A and 2 is assigned to B. At stage 1, same item i.e. 3 is preferred by both the players. If 3 is allocated to B then 4 has to be allocated to A. But that not EF because B has two items 2 and 3 that are more preferred by A than 4. If 3 is allocated to A, 5 has to be allocated to B but that is not EF too because here A contains two items 1 and 3 that are more preferred by B than 5. So 3 goes in CP. Now, 4 is allocated to A and 5 to B. At stage 3, remaining unallocated item is 6 which cannot be allocated to any player because the number of items allocated must be the same. So 6 also goes in CP. Thus, SA = { 1, 4 }, SB = { 2, 5 } and CP = { 3, 6 }. Coincidentally, both BT and AL produce the same allocation.

The AL allocation is the maximal EF allocation. No other EF allocation is possible with more number of items allocated than that by AL. Thus only AL gives EF allocations that are efficient or Pareto-optimal. A BT allocation is a locally Pareto-optimal (LPO): There is no other allocation of the items that each algorithm allocates that is at least as good as for A and B and better for one or more players. Because an AL allocation can allocate more or better items to one or both players, however it may globally Pareto-dominate the BT allocation.

6 Conclusion

Given that two players can rank the indivisible items from best to worst i.e. from most preferred to the least preferred, the AL algorithm can find an allocation of those items to the players that is fair, envy-free, Pareto-optimal and maximal. On the other hand, the BT algorithm, the simpler algorithm can allocate fewer items to the players, so it may not be maximal but it is locally Pareto-optimal. Even if it is maximal it is Pareto-dominated by the AL algorithm for the same problem and may produce better results. Thus both the algorithms have their advantages and disadvantages. In this way, the indivisible items can be allocated to two people fairly and envy-freely.

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References

[1] Sylvain Bouveret, Ulle Endriss, and Jerome Lang, Fair Division under Ordinal Preferences: Computing Envy-Free Allocations of Indivisible Goods, Proceedings of 2010 Conference on ECAI 2010: 19^{th} European Conference on Artificial Intelligence, vol. 215. Amsterdam: IOS Press, 2010, pp. 387-392.

[2] Steven J. Brams, Michael A. Jones, and Kristier Klamler, N-person cake cutting: There may be no perfect division, American Mathematical Monthly120, no. 1, January 2013, 35-47.

[3] Steven J. Brams and Alan D. Taylor, Fair Division: From Cake Cutting to Dispute Resolution, Cambridge, New York: Cambridge University Press, 1996.

[4] Charles Lumet, Sylvain Bouveret, and Michael Lemaitre, Fair Division of Indivisible Goods under Risk, Proceedings of the 20^{th} European Conference on Artificial Intelligence 2012, 564-569.

[5] Sylvain Bourevet and Jerome Lang, Efficiency and Envy-Freeness in Fair Division of Indivisible goods: logical representationa and complexity, Journal of Artificial Intelligence Research 32 (2008) 525-564.